DTU Library

Turbulence modelling

Sørensen, Niels N.

Publication date:
2010

Turbulence modelling

Glasgow - 2010

Niels N. Sørensen

Wind Energy Division · Risø DTU

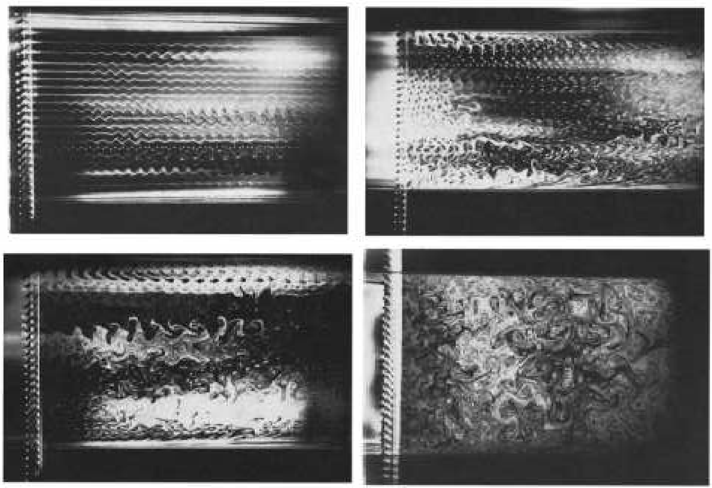
RISØ-DTU, 09-09-2010

Outline

The Nature of turbulence

- Irregularity
  - Turbulence is irregular or random.
- Diffusivity
  - Turbulent flow causes rapid mixing, increases heat transfer and flow resistance. These are the most important aspect of turbulence from a engineering point of view.
- Three-dimensional vorticity fluctuations (rotational)
  - Turbulence is rotational, and vorticity dynamics plays an important role. Energy is transferred from large to small scale by the interaction of vortices.
- Dissipation
  - Turbulent flow are always dissipative. Viscous shear stresses perform deformation work which increases the internal energy of the fluid at the expenses of kinetic energy of turbulence.
- Continuum
  - Even though they are small the smallest scale of turbulence are ordinary far lager than any molecular length scale
- Flow feature
  - Turbulence is a feature of the flow not of the fluid.

How Does Turbulence Look



The Onset of Two-Dimensional Grid Generated Turbulence in Flowing Soap Films
Maarten A. Rutgers, Xiao-lun Wu, and Walter I. Goldberg

Modeling Turbulent Flows, DNS

- Direct Numerical Simulation (DNS) of channel flow
  - All Scales of the fluid motion spatial and temporal are resolved by the computations. The largest DNS to date is 4096^3.

$$Re_\tau = \frac{u_\tau H/2}{\nu}.$$

$$N_{DNS}^3 \geq Re_\tau^{2.25}$$

Re_H	Re_τ	N_{DNS}^3	Timesteps
12.300	360	6.7×10^6	32.000
30.800	800	4.0×10^7	47.000
61.600	1.450	1.5×10^8	63.000
230.000	4.650	2.1×10^9	114.000

Niels N. Sørensen,
Risø DTU

Modeling Turbulent Flows, LES

- Large Eddy Simulation (LES) of channel flow
  - Only the large spatial and temporal scales are resolved by the computations

$$Re_\tau = \frac{u_\tau H/2}{\nu} .$$

$$N^3_{LES} \geq \left(\frac{0.4}{Re_\tau^{0.25}} \right) Re_\tau^{2.25}$$

Re_H	Re_τ	N^3_{DNS}	N^3_{LES}	Timesteps
12.300	360	6.7×10^6	6.1×10^5	3.2000
30.800	800	4.0×10^7	3.0×10^6	4.7000
61.600	1.450	1.5×10^8	1.0×10^7	6.3000
230.000	4.650	2.1×10^9	1.0×10^8	11.4000

Modeling Turbulent Flows, RANS

- Reynolds Averaged Navier-Stokes (RANS) of channel flow
  - The resolve the equations are time averaged and do not resolve the eddies

$$Re_\tau = \frac{u_\tau H/2}{\nu} .$$

Re_H	Re_τ	N^3_{DNS}	N^3_{LES}	N^3_{RANS}
230.000	4.650	2.1×10^9	1.0×10^8	1.0×10^4

- A hybrid model can be developed using LES/RANS

Niels N. Sørensen,
Risø DTU

Navier-Stokes equations

- Incompressible Navier-Stokes equation

The flow equations and additional equations have the following form:

Continuity equation:

$$\frac{\partial}{\partial x_j}(\rho U_j) = 0$$

Momentum equations:

$$\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{\partial x_j}(\rho\left(U_i U_j\right)) - \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right] + \frac{\partial P}{\partial x_i} = S_v \ ,$$

Auxiliary equations:

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x_j}(\rho(U_j\phi)) - \frac{\partial}{\partial x_j}\left[\mu\frac{\partial \phi}{\partial x_i}\right] = S_\phi$$

Niels N. Sørensen,
Risø DTU

Reynolds Averaged Navier-Stokes

- Reynolds averaging of the Navier-Stokes equation, splitting the velocities in the mean and the fluctuating component

$$u_i(\vec{r},t) = U_i(\vec{r}) + u'(\vec{r},t)\text{ , where } U_i(\vec{r}) = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} u_i(\vec{r},t)dt$$

- Inserting the Reynolds decomposed velocity in the Navier-Stokes and continuity equations
- Perform time averaging of the equations. The equations are in principle time independent, or steady state.

$$\overline{U_i(\vec{r}))} = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} U_i(\vec{r})dt = U_i(\vec{r})$$

$$\overline{u_i(r,t)'} = \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\left[u_i(\vec{r},t) - U_i(\vec{r})\right]dt = U_i(\vec{r}) - \overline{U_i(\vec{r})} = 0$$

Niels N. Sørensen,
Risø DTU

The Reynolds Averaged Navier-Stokes

The flow equations and additional equations have the following form:

Continuity equation:

$$\frac{\partial}{\partial x_j}(\rho U_j) = 0$$

Momentum equations:

$$\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{\partial x_j}\left(\rho \left(U_i U_j + u_i' u_j'\right)\right) - \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right] + \frac{\partial P}{\partial x_i} = S_v \ ,$$

Auxiliary equations:

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x_j}(\rho(U_j\phi + u_j'\phi')) - \frac{\partial}{\partial x_j}\left[\mu\frac{\partial\phi}{\partial x_i}\right] = S_\phi$$

Niels N. Sørensen,
Risø DTU

Reynolds Stresses

- Performing the Reynolds Averaging Process, new terms has arisen, namely the Reynolds stress tensor:

$$\tau_{ij} = -\overline{\rho u_i' u_j'}$$

- This brings us at the turbulent closure problem, the fact that we have more unknowns than equations
  - Three velocities+pressure+six Reynolds-stresses
  - Three momentum-equations+continuity equation
- To close the problem, we need additional equations to model the Reynolds-stresses

Niels N. Sørensen,
Risø DTU

The Reynolds Averaged Momentum-equations

- The Reynold Stresses originates from the convective terms

$$\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{\partial x_j}(\rho \left(U_i U_j + u_i' u_j'\right)) - \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right] + \frac{\partial P}{\partial x_i} = S_v \ ,$$

- The Reynold Stresses are often treated together with the diffusive terms

$$\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{\partial x_j}(\rho \left(U_i U_j\right)) - \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - u_i' u_j'\right] + \frac{\partial P}{\partial x_i} = S_v \ ,$$

Niels N. Sørensen,
Risø DTU

Reynold-stress equations

Performing the following operation on the Navier-Stokes equation, equations for the Reynold-stresses can be derived:

$$\overline{u_i' NS(u_i) + u_j' NS(u_i)} = 0$$

$$\frac{\partial \tau_{ij}}{\partial t} + U_k \frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{jk}\frac{\partial U_i}{\partial x_k} + 2\mu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} + \overline{u_i'\frac{\partial p'}{\partial x_j} + u_j'\frac{\partial p'}{\partial x_j}} + \frac{\partial}{\partial x_k}\left[\nu\frac{\partial \tau_{ij}}{\partial x_k} + \rho\overline{u_i' u_j' u_k'}\right]$$

- The procedure introduces new unknowns (22 new unknowns)

$$\overline{\rho u_i' u_j' u_k'} \rightarrow 10 \text{ unknowns}$$

$$2\mu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \rightarrow 6 \text{ unknowns}$$

$$\overline{u_i'\frac{\partial p'}{\partial x_j} + u_j'\frac{\partial p'}{\partial x_j}} \rightarrow 6 \text{ unknowns}$$

Niels N. Sørensen,
Risø DTU

Boussinesq Eddy Viscosity Approximation

- Eddy-viscosity models
  - Compute the Reynolds-stresses from explicit expressions of the mean strain rate and a eddy-viscosity, **the Boussinesq eddy-viscosity approximation**

$$\tau_{ij} = \rho\overline{u_i' u_j'} = -2\mu_t S_{ij} - \frac{2}{3}\rho k \delta_{ij} \ ,$$

$$, \text{where } S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \text{ , and } -2\rho k = \tau_{ii} = -\overline{\rho u_i' u_j'} \ .$$

- The k term is a normal stress and is typically treated together with the pressure term

Niels N. Sørensen,
Risø DTU

The Reynolds Averaged Navier-Stokes

The flow equations and additional equations have the following form:

Continuity equation:

$$\frac{\partial}{\partial x_j}(\rho U_j) = 0$$

Momentum equations:

$$\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{\partial x_j}(\rho U_i U_j) - \frac{\partial}{\partial x_j}\left[(\mu - \mu_t)\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right] + \frac{\partial \hat{P}}{\partial x_i} = S_v \ ,$$

Auxiliary equations:

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x_j}(\rho U_j \phi) - \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\phi}\right)\frac{\partial \phi}{\partial x_i}\right] = S_\phi$$

Niels N. Sørensen,
Risø DTU

RANS turbulence models

- Algebraic turbulence models
  - Prandtl Mixing Length Modeling
  - Cebeci-Smith Modeling
  - Baldwin-Lomax Model
- One equations turbulence models
  - Spalart-Allmaras
  - Baldwin-Barth
- Two equation turbulence models
  - $k - \epsilon$ model
  - $k - \omega$ model
  - $k - \tau$ models
- Reynolds stress models

Niels N. Sørensen,
Risø DTU

Algebraic Turbulence Model

- Prandtls mixing length hypothesis is based on an analogy with momentum transport on a molecular level

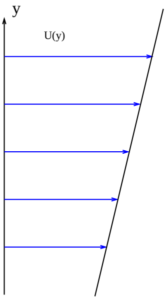
- Molecular transport

$$t_{xy} = \mu \frac{dU}{dy} \text{ , where } \mu = \frac{1}{2}\rho v_{th} l_{fmp}$$

- Turbulent transport

$$\tau_{xy} = \mu_t \frac{dU}{dy} \text{ , where } \mu_t = \frac{1}{2}\rho v_{mix} l_{mix}$$

and, $v_{mix} = c_1 l_{mix} \left| \frac{dU}{dy} \right|$, and $l_{mix} = c_2 y$



Niels N. Sørensen,
Risø DTU

Prandtl Mixing Length Model

- The mixing length model closes the equations system
- Turbulent transport

$$\tau_{xy} = \mu_t \frac{dU}{dy} \text{, where } \mu_t = \frac{1}{2}\rho v_{mix} l_{mix}$$

$$\text{and, } v_{mix} = c_1 l_{mix} \left| \frac{dU}{dy} \right| \text{, and } l_{mix} = c_2 y$$

- The proportionality constant for the mixing velocity c_1 and for the mixing length c_2 needs to be specified
- The equation for the turbulent eddy viscosity is a part of the flow solutions, as it depends on the mean flow gradient
- turbulence is not a fluid property but a property of the flow

Niels N. Sørensen,
Risø DTU

Algebraic turbulence models

Baldwin-Lomax Model

$$\mu_t = \begin{cases} \mu_{inner} = \rho l^2 \left|\Omega\right| & \text{if } \; y \leq y_{cross} \\ \mu_{outer} = \rho K C_{wp} F_{wake} F_{Kleb}(y) & \text{if } \; y > y_{cross} \end{cases} .$$

$$y_{cross} = \min(y) \text{ where } \mu_{outer} = \mu_{inner}$$

$$\left|\Omega\right| = \sqrt{2\Omega_{ij}\Omega_{ij}} \text{ , and } \Omega_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) \text{ , and } l = \kappa y \left(1 - e^{\frac{-y^+}{A^+}}\right)$$

$$F_{wake} = \min\left(y_{max} F_{max}, C_{wk} y_{max} \frac{u_{diff}^2}{F_{max}}\right) \text{ , and } F_{Kleb} = \left[1 + 5.5\left(\frac{y C_{Kleb}}{y_{max}}\right)^6\right]^{-1}$$

$$\text{, and } u_{diff} = \max\left(\sqrt{U_i U_k}\right) - \min\left(\sqrt{U_i U_k}\right)$$

y_{max} and F_{max} is determined by the maximum of $F(y) = y \left|\Omega\right| \left(1 - e^{\frac{y^+}{A^+}}\right)$

$$A^+ = 26 \text{ , } C_{cp} = 1.6 \text{ , } C_{Kleb} = 0.3 \text{ , } C_{wk} = 0.25 \text{ , } \kappa = 0.4 \text{ , } K = 0.0168$$

Niels N. Sørensen,
Risø DTU

Algebraic Models

- Gives good results for simple flows, flat plate, jets, simple shear layers and airfoils at low AOA
- Typically the algebraic models are fast and robust
- Needs to be calibrated for each flow type, they are not very general
- They are not well suited for computing flow separation
- Typically they need information about the boundary layer properties and wake location, and are difficult to incorporate in modern flow solvers.

One and Two Equation Turbulence Models

- The derivation is again based on the Boussinesq approximation

$$\tau_{xy} = \mu_t \frac{dU}{dy} \text{ , where } \mu_t = \frac{1}{2}\rho v_{mix} l_{mix}$$

- The mixing velocity is determined by the turbulent kinetic energy

$$v_{mix} \sim k^{\frac{1}{2}} \text{ , and } k = \frac{1}{2}\overline{u_i' u_j'}$$

- The mixing length is determined from an algebraic relation or using another transport equation

$$l_{mix} \sim \frac{k^{\frac{3}{2}}}{\epsilon} \text{ , where } \epsilon = \nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}}$$

Niels N. Sørensen,
Risø DTU

Transport equation based turbulence models

Second equation

Proposer(s)	(year)	Variable	Symbol
Kolmogorov	(1942)	$k^{\frac{1}{2}}/l$	ω
Saffman	(1970)		
Wilcox et al.	(1972)		
Chou	(1945)	$k^{\frac{3}{2}}/l$	ϵ
Davidov	(1961)		
Harlow-Nakayama	(1968)		
Jones-Launder	(1972)		
Rotta	(1951)	l	l
Rotta	(1968,1971)	kl	kl
Rodi-Spalding	(1970)		
Ng-Spalding	(1972)		
Spalding	(1969)	k/l^2	W

Niels N. Sørensen,
Risø DTU

The k-equation

- By taking the trace of the Reynolds Stress equation we get

$$\rho\frac{\partial k}{\partial t}+\rho U_j\frac{\partial k}{\partial x_j}=\tau_{ij}\frac{\partial U_i}{\partial x_j}-\rho\epsilon+\frac{\partial}{\partial x_j}\left[\mu\frac{\partial k}{\partial x_j}-\frac{1}{2}\overline{\rho u_i'u_i'u_j'}-\overline{p'u_j'}\right]$$

where $k=\frac{1}{2}\overline{u_i'u_i'}$ and $\epsilon=\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}$

Using that $\frac{1}{2}\overline{\rho u_i'u_i'u_j'}-\overline{p'u_i'}=-\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_j}$ assuming $\overline{u_i'\phi'}\sim\mu_t\frac{\partial\phi}{\partial x_j}$ we get

$$\rho\frac{\partial k}{\partial t}+\rho U_j\frac{\partial k}{\partial x_j}=\tau_{ij}\frac{\partial U_i}{\partial x_j}-\rho\epsilon+\frac{\partial}{\partial x_j}\left[\left(\mu+\frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

Niels N. Sørensen,
Risø DTU

Dissipation of turbulent kinetic energy

- The equation for the dissipation of turbulent kinetic energy is derived by the following operation on the Navier-Stokes equations.

$$2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial}{\partial x_j} NS(u_i)} = 0$$

$$\rho \frac{\partial \epsilon}{\partial t} + \rho U_j \frac{\partial \epsilon}{\partial x_j} = -2\mu \frac{\partial U_i}{\partial x_j} \left[\overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}} + \overline{\frac{\partial u_k'}{\partial x_i} \frac{\partial u_k'}{\partial x_j}} \right] + 2\mu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_m} \frac{\partial u_k'}{\partial x_m}}$$

$$-2\mu \overline{\left[\frac{\partial^2 u_i'}{\partial x_k} \partial x_m \right]^2} - \frac{\partial}{\partial x_j} \left[\overline{u_j \epsilon} + \nu \overline{\frac{\partial p'}{\partial x_m} \frac{\partial u_j'}{\partial x_m}} \right]$$

Niels N. Sørensen,
Risø DTU

The $k - \epsilon$ model

The standard $k - \epsilon$ model is only applicable for high Reynolds numbers, and therefore needs wall modelling.

- Eddy viscosity

$$\mu_t = \rho C_\mu \frac{k^2}{\epsilon}$$

- Transport equation for turbulent kinetic energy

$$\rho \frac{\partial k}{\partial t} + \rho U_j \frac{\partial k}{\partial x_j} = \tau_{ij} \frac{\partial U_i}{\partial x_j} - \rho \epsilon + \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right]$$

- Transport equation for dissipation of turbulent kinetic energy

$$\rho \frac{\partial \epsilon}{\partial t} + \rho U_j \frac{\partial \epsilon}{\partial x_j} = C_{\epsilon 1} \frac{\epsilon}{k} \tau_{ij} \frac{\partial U_i}{\partial x_j} - \rho C_{\epsilon 2} \frac{\epsilon^2}{k} + \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_t}{\sigma_\epsilon} \right) \frac{\partial \epsilon}{\partial x_j} \right]$$

$$C_{\epsilon 1} = 1.33 \ , \ C_{\epsilon 2} = 1.92 \ , \ C_\mu = 0.09 \ , \ \sigma_k = 1.0 \ , \ \sigma_\epsilon = 1.0 \ ,$$

Niels N. Sørensen,
Risø DTU

The $k - \epsilon$ model, low Reynolds number version

$$\mu_t = \rho C_\mu f_\mu \frac{k^2}{\epsilon}$$

$$\rho \frac{\partial k}{\partial t} + \rho U_j \frac{\partial k}{\partial x_j} = \tau_{ij} \frac{\partial U_i}{\partial x_j} - \rho \epsilon + \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] - 2\mu \left(\frac{\partial k^{\frac{1}{2}}}{\partial x_j} \right)^2$$

- Transport equation for dissipation of turbulent kinetic energy

$$\rho \frac{\partial \overline{\epsilon}}{\partial t} + \rho U_j \frac{\partial \overline{\epsilon}}{\partial x_j} = C_{\epsilon 1} f_1 \frac{\overline{\epsilon}}{k} \tau_{ij} \frac{\partial U_i}{\partial x_j} - \rho C_{\epsilon 2} f_2 \frac{\overline{\epsilon}^2}{k} + E + \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\mu_t}{\sigma_\epsilon} \right) \frac{\partial \overline{\epsilon}}{\partial x_j} \right]$$

where $\overline{\epsilon} = \epsilon - \epsilon_0$ and ϵ_0 is the value of ϵ at the wall and boundary conditions $k = \epsilon_0 = 0$.

Niels N. Sørensen,
Risø DTU

The $k - \omega$ model

The $k - \omega$ model of Menter is a blend of the original $k - \omega$ model of Wilcox, the $k - \epsilon$ model and the Bradshaw et al. model.

- Eddy viscosity:

$$\mu_t = \frac{\rho a_1 k}{\max(a_1\omega; F_2\Omega)}$$

The model is tuned through the F_2 function to switch to the Bradshaw assumption in adverse pressure gradients

$$\mu_t = \frac{\tau}{\Omega} = \rho\frac{a_1 k}{\Omega}, \; a_1 = 0.30$$

The $k-\omega$ model

- Eddy viscosity

$$\mu_t = \frac{\rho a_1 k}{\max(a_1\omega; F_2\Omega)}$$

- Transport equation for turbulent kinetic energy

$$\rho\frac{\partial k}{\partial t} + \rho U_j \frac{\partial k}{\partial x_j} = \tau_{ij}\frac{\partial U_i}{\partial x_j} - \rho\beta^* k\omega + \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]$$

- Transport equation for the specific dissipation rate

$$\rho\frac{\partial \omega}{\partial t} + \rho U_j \frac{\partial \omega}{\partial x_j} = \frac{\gamma}{\nu_t}\tau_{ij}\frac{\partial U_i}{\partial x_j} - \beta\rho\omega^2 + 2\rho(1-F_1)\frac{1}{\sigma_{\omega 2}\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_j}\right]$$

Blending Function's F_1 and F_2

$$F_1 = \tanh\left(arg_1^4\right) \text{ and } F_2 = \tanh\left(arg_2^4\right)$$

where

$$arg_1 = \min\left[\max\left(\frac{\sqrt{k}}{0.09\omega y}; \frac{500\mu}{y^2\omega}\right); \frac{4\rho k}{CD_{k\omega} y^2 \sigma_{\omega 2}}\right] ,$$

$$arg_2 = \max\left(2\frac{\sqrt{k}}{0.09\omega y}; \frac{500\mu}{y^2\omega}\right)^2 ,$$

and $CD_{k\omega}$ represents the cross-diffusion term

$$CD_{k\omega} = \max\left(2\rho\frac{1}{\sigma_{\omega 2}\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}; 10^{-20}\right) .$$

Constants for the $k-\omega$ SST model

- Constants for the inner region ($k-\omega$)

β_1	β^*	γ_1	σ_{k1}	$\sigma_{\omega 1}$	a_1
0.075	0.090	0.553	1.176	2.000	0.310

- Constants for the outer region ($k-\epsilon$)

β_2	β^*	γ_2	σ_{k2}	$\sigma_{\omega 2}$	a_1
0.828	0.090	0.4404	1.000	1.170	0.310

- The actual constant is obtained by blending the inner and outer constants

$$\alpha = \alpha_{inner} F_1 + \alpha_{outer}(1 - F_1)$$

Niels N. Sørensen,
Risø DTU

Wall boundary conditions for the $k - \omega$ model

- The $k - \omega$ model is valid all the way to the wall, and do not need low Re modifications
- The boundary conditions are very simple to apply.

At the wall we have

$$k = 0 \text{ , and } \omega = 10 \frac{6\nu}{\beta_1 \Delta y^2}$$

The velocity boundary conditions is a simple no-slip condition.

- The model is robust in the low Re version, and only demands $y^+ \sim 2$

Niels N. Sørensen,
Risø DTU

Farfield boundary conditions for the $k - \omega$ model

- Typically, the inflow turbulence intensity is known

$$k = IU_\infty^2$$

- For many aerodynamic applications, like airfoil computations, inflow is often nearly laminar in the farfield $\nu_t << \nu$
- This can be approximated by $\nu_t = 1 \times 10^{-3} \nu$
- This leads to

$$\omega = C_\mu \frac{k}{\nu_t} = C_\mu 1 \times 10^3 \frac{k}{\nu}$$

For wall bounded cases, the inflow value of ω can often be computed using the mixing length hypothesis assuming a velocity profile.

Niels N. Sørensen,
Risø DTU

Large Eddy Simulation

Filtering of the Navier-Stokes equation, splitting the velocities in the resolvable-scale or filtered velocity and the sub-grid scale (SGS) velocity

$$u_i = \overline{u_i} + u'$$

$$\overline{u_i}(\vec{r}, t) = \int\int\int G(\vec{r} - \vec{\xi}; \Delta) u_i(\vec{\xi}, t) d^3\vec{\xi}$$

with

$$\int\int\int G(\vec{r} - \vec{\xi}; \Delta) d^3\vec{\xi} = 1$$

A typical filter could be the volume-averaged box filter

$$G(\vec{r} - \vec{\xi}; \Delta) = \begin{cases} 1/\Delta^3 & , |\vec{r}_i - \vec{\xi}_i| \\ 0 & \text{, otherwise} \end{cases}$$

Niels N. Sørensen,
Risø DTU

LES filtering of the Navier-Stokes equations

- As with the Reynolds Averaging procedure, the filtering generates new terms from the convective terms

$$\overline{u_i u_j} = \overline{u_i}\,\overline{u_i} + L_{ij} + C_{ij} + R_{ij}$$

where

$$L_{ij} = \overline{\overline{u_i}\,\overline{u_j}} - \overline{u_i}\,\overline{u_j}$$

$$C_{ij} = \overline{\overline{u_i} u_j'} + \overline{\overline{u_j} u_j'}$$

$$R_{ij} = \overline{u_i' u_j'}$$

- Filtering differs from standard averaging in one important respect

$$\overline{\overline{u_i}} \neq \overline{u_i}$$

Niels N. Sørensen,
Risø DTU

LES and DES modeling

frame

- The Leonard stresses (L) are of the same order as the typical truncation error in a second-order accurate scheme
- The cross-term stress tensor (C) are typically modeled together with the Reynolds stresses

$$Q_{ij} = R_{ij} + C_{ij}$$

- The first model for the sub-grid scale stresses (SGS) was the model by Smagorinsky (1963) based on the gradient diffusion approximation

$$\tau_{ij} = 2\mu_t S_{ij} \text{ , where } S_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right)$$

$$\mu_t = \rho C_s^2 \Delta^2 \sqrt{S_{ij}S_{ij}} \text{ , and } C_s = [0.10 : 0.24]$$

LES filtering of the Navier-Stokes equations

- Smagorinsky model

$$\frac{\partial}{\partial t}\left(\rho \overline{u_i}\right) + \frac{\partial}{\partial x_j}\left(\rho \overline{u_i}\,\overline{u_j}\right) = \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right) - \tau_{ij}\right] - \frac{\partial P}{\partial x_j} + S_v$$

$$\tau_{ij} = -\rho\left(Q_{ij} - \frac{1}{3}Q_{kk}\delta_{ij}\right) = 2\mu_t S_{ij}$$

$$P = \overline{p} + \frac{1}{3}Q_{kk}\delta_{ij}$$

$$Q_{ij} = R_{ij} + C_{ij}$$

$$\mu_t = \rho(C_s\Delta)^2\sqrt{S_{ij}S_{ij}}\ \text{, and } C_s \sim [0.10 : 0.24]$$

Niels N. Sørensen,
Risø DTU

LES modeling

- LES models are by nature unsteady
- LES models are by nature fully three dimensional
- They resolve the large scales and only model the isotropic small scales
- The standard SGS model needs damping of the eddy viscosity near solid walls similar to the van Driest damping
- Resolving the anisotropic eddies in the near wall region where the scale are small may require a very fine grid
- Pure LES models are to computational demanding for most airfoil and rotor computations
- LES models can be combined with approximate wall boundary conditions, or even zero, one or two equation models for the region near walls

Hybrid models

- The original DES model of Spallart et al. based on the Spallart-Allmaras model
- The $k-\omega$ DES model
- The $k-\omega$ DDES model
- The SAS model

Niels N. Sørensen,
Risø DTU

DES and DDES modeling

The idea behind the DES modeling is to exchange the turbulent length scale with the grid size when the turbulent length scale become larger than the grid size, and the grid is capable of resolving some of the scales.

- The turbulent length scale in the $k - \omega$ model is given by

$$L_t^{k-\omega} = \frac{k^{\frac{3}{2}}}{\epsilon} = \frac{\sqrt{k}}{\beta^* \omega} \text{ using } \epsilon = \beta^* k \omega$$

- The turbulent length scale in a LES would be

$$L_t^{LES} = \Delta C_{Des} \text{ with } \Delta = \min\left[\Delta x, \Delta y, \Delta z\right]$$

- To enforce the L_t^{LES} when the grid allows, we will simply scale the dissipation term in the ω equation by the ratio between $L_t^{k-\omega}$ and L_t^{LES}.

$$\beta^* \omega k = \beta^* \omega k \max\left(\frac{L_t^{k-\omega}}{L_t^{DES}} F_{Shield}; 1\right)$$

Niels N. Sørensen,
Risø DTU

DES and DDES modeling

The idea behind the DES modeling is to exchange the turbulent length scale with the grid size when the turbulent length scale become larger than the grid size, and the grid is capable of resolving some of the scales.

- The turbulent length scale in the $k - \omega$ model is given by

$$L_t^{k-\omega} = \frac{k^{\frac{3}{2}}}{\epsilon} = \frac{\sqrt{k}}{\beta^* \omega} \text{ using } \epsilon = \beta^* k \omega$$

- The turbulent length scale in a LES would be

$$L_t^{LES} = \Delta C_{Des} \text{ with } \Delta = \min\left[\Delta x, \Delta y, \Delta z\right]$$

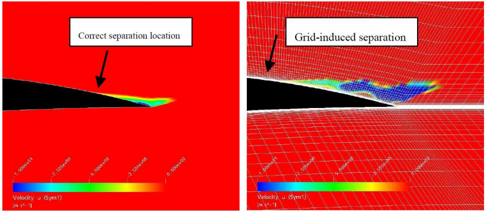
- To enforce the L_t^{LES} when the grid allows, we will simply scale the dissipation term in the ω equation by the ratio between $L_t^{k-\omega}$ and L_t^{LES}.

$$\beta^* \omega k = \beta^* \omega k \max\left(\frac{L_t^{k-\omega}}{L_t^{DES}}(1 - F_{Shield}); 1\right)$$

Niels N. Sørensen,
Risø DTU

DES and DDES modeling

- Grid Induced Separation, or Modeled Stress Depletion (MSD)
  - One should avoid that the LES part of the model is active within the boundary layer



- The boundary layer can be shielded by a damping function eg. the F_1 or the F_2 blending functions from the $k - \omega$ SST model

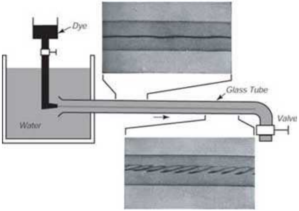
Niels N. Sørensen,
Risø DTU

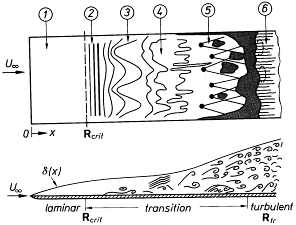
Laminar turbulent transition

Often the flow is not fully turbulent, this was already noticed by Osbourne Reynolds in 1883

- The transition process depend on many parameters
  - Reynolds Number
  - Free stream turbulence level
  - Laminar separation bubbles
  - Cross flow
  - Surface Roughness
  - Mass injection

- Typically approaches for transition modeling
  - e^n method (Orr-Sommerfeld eqn.)
  - Empirical Correlations
    - Michel
    - Mayle
    - Abu-Ghannam and Shaw
    - Suzen





Niels N. Sørensen,
Risø DTU

The $\gamma - Re_\theta$ Correlation based transition model

- The model is based on comparing the local Momentum Thickness Reynolds number with a critical value from empirical expressions

$$Re_\theta = Re_{\theta t}$$

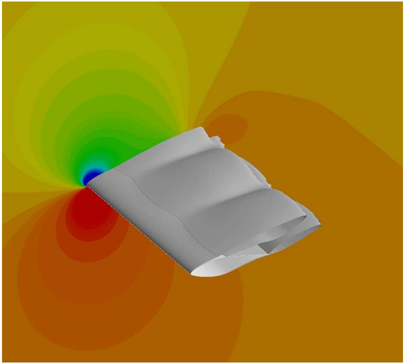
- The following relation is used to simplify the computations in a general CFD code

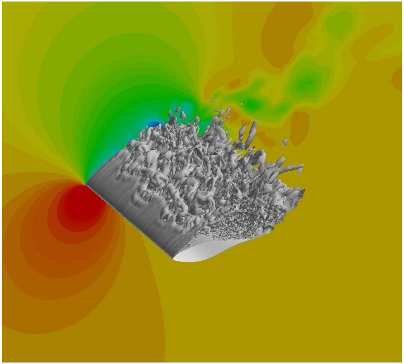
$$Re_\theta = \frac{Re_{\nu_t max}}{2.193}$$

- The model is based on transport equations, and can easily be implemented in general purpose flow solvers
- In the present form the model handles natural transition, by-pass transition, and separation induced transition
- The transition model is coupled to the $k - \omega$ SST model through the production and destruction terms in the k-equation

Conclusion

- I have talked briefly about the basic property of turbulence
- The boussinesq approximation has been given
- The RANS, LES and DES methods has been introduced
- A few popular modelling approaches has been shown





Niels N. Sørensen,
Risø DTU